

A new class of interpolatory L -splines with adjoint end conditions

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Abstract. A thin plate spline for interpolation of smooth transfinite data prescribed along concentric circles was recently proposed by Bejancu, using Kounchev's polyspline method. The construction of the new 'Beppo Levi polyspline' surface reduces, via separation of variables, to that of a countable family of univariate L -splines, indexed by the frequency integer k . This paper establishes the existence, uniqueness and variational properties of the 'Beppo Levi L -spline' schemes corresponding to non-zero frequencies k . In this case, the resulting L -spline end conditions are formulated in terms of *adjoint* differential operators, unlike the usual 'natural' L -spline end conditions, which employ identical operators at both ends. Our L -spline error analysis leads to an L^2 -error bound for transfinite surface interpolation with Beppo Levi polysplines.

Keywords: interpolation, L -spline, Beppo Levi polyspline, approximation order

1 Introduction

The *thin plate spline* (TPS) interpolant for scattered data was defined by Duchon [11] as the unique minimizer of the squared seminorm

$$\|F\|_{BL}^2 := \iint_{\mathbb{R}^2} \left(|F_{xx}|^2 + 2|F_{xy}|^2 + |F_{yy}|^2 \right) dx dy, \quad (1)$$

subject to F taking prescribed values at a finite number of scattered locations. The minimization takes place in the *Beppo Levi* space of continuous functions F with generalized second-order partial derivatives in $L^2(\mathbb{R}^2)$.

Recently, Bejancu [5] proposed a new type of TPS surface, passing through several continuous curves prescribed along concentric circles. The new surface minimizes, for $F \in C^2(\mathbb{R}^2 \setminus \{0\})$, the polar coordinate version of (1):

$$\|f\|_{BL}^2 := \int_0^\infty \int_{-\pi}^\pi \left\{ |f_{rr}|^2 + 2 \left| \frac{f_\theta}{r^2} - \frac{f_{\theta r}}{r} \right|^2 + \left| \frac{f_{\theta\theta}}{r^2} + \frac{f_r}{r} \right|^2 \right\} r d\theta dr, \quad (2)$$

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where $f(r, \theta) := F(r \cos \theta, r \sin \theta)$ denotes the polar form of F . Similar surfaces for *transfinite* interpolation have also been studied in [3,4] in the case of continuous periodic data prescribed along parallel lines or hyperplanes (see also the survey [6]).

The ‘transfinite TPS’ surfaces belong to the class of multivariate *polysplines* introduced by Kounchev [14]. In the context of data prescribed on concentric circles $r = r_j$, $j \in \{1, \dots, n\}$, with $0 < r_1 < \dots < r_n$, let us denote $\rho := \{r_1, \dots, r_n\}$ and $\Omega := \{(r, \theta) : r_1 \leq r \leq r_n, -\pi \leq \theta \leq \pi\}$. A function $S : \Omega \rightarrow \mathbb{R}$ is termed a *biharmonic polyspline* on annuli determined by ρ if two conditions hold: first, S and its polar form s are piecewise biharmonic, *i.e.*

$$(\partial_{xx} + \partial_{yy})^2 S(x, y) = (\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta})^2 s(r, \theta) = 0,$$

on each annulus $r_j < r < r_{j+1}$, $-\pi \leq \theta \leq \pi$, for $1 \leq j \leq n-1$; and second, $S \in C^2(\Omega)$, *i.e.* neighbouring pieces join up C^2 -continuously across the interface circles. For sufficiently smooth periodic data functions $u, v, \mu_j : [-\pi, \pi] \rightarrow \mathbb{R}$, $1 \leq j \leq n$, Kounchev proved that such a polyspline surface is uniquely determined by transfinite interpolation conditions

$$s(r_j, \theta) = \mu_j(\theta), \quad \forall \theta \in [-\pi, \pi], \quad \forall j \in \{1, \dots, n\}, \quad (3)$$

together with boundary conditions $\partial_r s(r_1, \theta) = u(\theta)$ and $\partial_r s(r_n, \theta) = v(\theta)$, $\forall \theta \in [-\pi, \pi]$. He also extended this result to polysplines of higher orders and more general interface configurations in \mathbb{R}^d . In the case of *cardinal* interpolation at the bi-infinite set of hyperspheres of radii $r = e^j$, $j \in \mathbb{Z}$, Kounchev and Render [16] constructed polysplines that satisfy growth conditions as $r \rightarrow 0$ and $r \rightarrow \infty$.

In [5], Bejancu proposed a global polyspline $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ for which boundary conditions on the above extreme circles $r = r_1$ and $r = r_n$ are replaced by the requirement that the polar Beppo Levi energy (2) is finite for $f := s$. This *Beppo Levi polyspline* has two additional biharmonic pieces over the extreme annuli $0 < r < r_1$ and $r > r_n$, such that $S \in C^2(\mathbb{R}^2 \setminus \{0\})$. The new surface is automatically continuous at 0, but its partial derivatives can have a singularity at 0.

For sufficiently smooth data, it turns out that there exists a one-parameter family of such Beppo Levi polysplines on annuli determined by ρ , each satisfying the transfinite interpolation conditions (3). Two surfaces S^A and S^B of this family are uniquely determined in [5, Theorem 1] by the following additional conditions: S^A takes an arbitrarily prescribed value at 0, while S^B is biharmonic at 0 (hence, non-singular). Both S^A and S^B are then characterized as genuine TPS surfaces, *i.e.* minimizers of (2), subject to their respective interpolation conditions.

Following the method of separation of variables used by Kounchev [14], the construction of the Beppo Levi polysplines S^A, S^B is obtained in [5, section 4] via the absolutely convergent Fourier representation in polar form

$$s(r, \theta) = \sum_{k \in \mathbb{Z}} \hat{s}_k(r) e^{ik\theta}, \quad (r, \theta) \in [0, \infty) \times [-\pi, \pi]. \quad (4)$$

For each frequency k , the amplitude coefficient \widehat{s}_k of this representation is a univariate L_k -spline for an ordinary differential operator L_k , as described in the next section. Moreover, the form of \widehat{s}_k on the extreme intervals $(0, r_1)$ and (r_n, ∞) is determined by the condition that the corresponding Plancherel component of the Beppo Levi energy (2) of s is finite.

The present paper studies the class of such *Beppo Levi L_k -splines* corresponding to non-zero frequencies k (see section 2). In this case, the restrictions satisfied by \widehat{s}_k on the extreme intervals exhibit a twisted symmetry, expressed in terms of adjoint differential operators. Different features appear in the radial case $k = 0$, treated in the companion paper [7], which is connected to Rabut's work on radially symmetric thin plate splines [18].

In section 3, we prove the existence, uniqueness and variational characterization of interpolation schemes with Beppo Levi L_k -splines, as required by the construction of [5]. A part of these results, corresponding to $|k| \geq 2$, has first been obtained in the MSc thesis [2]. For $|k| \geq 2$, we also provide a linear representation of Beppo Levi L_k -splines in terms of dilates of a basis function related to the generalized Whittle-Matérn-Sobolev kernels introduced by Bozzini, Rossini, and Schaback [9]. Further, in section 4, we apply an error analysis of the Beppo Levi L_k -spline schemes to establish the L^2 -approximation order $O(h^2)$ for transfinite surface interpolation with Beppo Levi polysplines on annuli, where h is the maximum distance between successive interface circles. The extension of this work to higher order Beppo Levi polysplines on annuli and their L -spline Fourier coefficients will be addressed in a separate paper.

2 Preliminaries

2.1 Energy spaces

For each $r \geq 0$, define the Fourier coefficients of $f(r, \theta)$ with respect to θ by

$$\widehat{f}_k(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} f(r, \theta) d\theta, \quad k \in \mathbb{Z}. \quad (5)$$

The following observation shows the effect on \widehat{f}_k of the condition that the polar Beppo Levi integral (2) is finite. Namely, if f is the polar form of $F \in C^2(\mathbb{R}^2 \setminus \{0\})$, then $\widehat{f}_k \in C^2(0, \infty)$ and Plancherel's formula implies the identity [5, (3.2)]:

$$\|f\|_{BL}^2 = 2\pi \sum_{k \in \mathbb{Z}} \|\widehat{f}_k\|_k^2, \quad (6)$$

where, for each $k \in \mathbb{Z}$, we denote

$$\|\psi\|_k^2 := \int_0^\infty \left\{ \left| \frac{d^2\psi}{dr^2} \right|^2 + 2k^2 \left| \frac{\psi}{r^2} - \frac{1}{r} \frac{d\psi}{dr} \right|^2 + \left| k^2 \frac{\psi}{r^2} - \frac{1}{r} \frac{d\psi}{dr} \right|^2 \right\} r dr. \quad (7)$$

Let $AC_{\text{loc}}(0, \infty)$ be the vector space of functions $\psi : (0, \infty) \rightarrow \mathbb{C}$ that are absolutely continuous on any interval $[a, b]$, $0 < a < b < \infty$. We denote by

Λ_1 the vector space of functions $\psi \in C^1(0, \infty)$ with $\psi' \in AC_{\text{loc}}(0, \infty)$, such that $r^{1/2}\psi''$ and $r^{-1/2}\psi' - r^{-3/2}\psi$ belong to $L^2(0, \infty)$. Also, by Λ_2 we denote the vector space of functions $\psi \in C^1(0, \infty)$ with $\psi' \in AC_{\text{loc}}(0, \infty)$, such that $r^{1/2}\psi''$, $r^{-1/2}\psi'$, and $r^{-3/2}\psi$ all belong to $L^2(0, \infty)$. Note that $\|\cdot\|_k$ is a norm on Λ_2 for $|k| \geq 2$ and a semi-norm on Λ_1 for $k = \pm 1$. The results of section 3 employ the following properties of functions from the spaces Λ_1 and Λ_2 .

Lemma 1. (i) If $\psi \in \Lambda_2$, there exist non-negative constants C_ψ and \tilde{C}_ψ , such that

$$\begin{aligned} |\psi(r)| &\leq C_\psi \left(r^{3/2} + r|1-r|^{1/2} \right), \\ |\psi'(r)| &\leq \tilde{C}_\psi \left(r^{1/2} + |1-r|^{1/2} \right), \end{aligned} \quad \forall r > 0. \quad (8)$$

(ii) If $\psi \in \Lambda_1$, there exist non-negative constants C_ψ and \tilde{C}_ψ , such that

$$\begin{aligned} |\psi(r)| &\leq C_\psi r \left(1 + |\ln r|^{1/2} \right), \\ |\psi'(r)| &\leq \tilde{C}_\psi \left(1 + |\ln r|^{1/2} \right), \end{aligned} \quad \forall r > 0. \quad (9)$$

Proof. (i) For each $r > 0$, we use the Leibniz-Newton formula

$$\begin{aligned} r^{-3/2}\psi(r) - \psi(1) &= \int_1^r \left[t^{-3/2}\psi(t) \right]' dt \\ &= \int_1^r \left[t^{-3/2}\psi'(t) - \frac{3}{2}t^{-5/2}\psi(t) \right] dt. \end{aligned}$$

Via Cauchy-Schwarz, the last integral is bounded above in modulus by

$$\begin{aligned} &\left| \int_1^r t^{-1} \left[t^{-1/2}\psi'(t) \right] dt \right| + \frac{3}{2} \left| \int_1^r t^{-1} \left[t^{-3/2}\psi(t) \right] dt \right| \\ &\leq \left| \int_1^r t^{-2} dt \right|^{\frac{1}{2}} \left\{ \left| \int_1^r \left| t^{-1/2}\psi'(t) \right|^2 dt \right|^{\frac{1}{2}} + \frac{3}{2} \left| \int_1^r \left| t^{-3/2}\psi(t) \right|^2 dt \right|^{\frac{1}{2}} \right\} \\ &\leq |1-r^{-1}|^{\frac{1}{2}} \left\{ \left\| r^{-1/2}\psi' \right\|_{L^2(0,\infty)} + (3/2) \left\| r^{-3/2}\psi \right\|_{L^2(0,\infty)} \right\} \end{aligned}$$

which implies the first of inequalities (8). For the second inequality, we similarly start with

$$\begin{aligned} r^{-1/2}\psi'(r) - \psi'(1) &= \int_1^r \left[t^{-1/2}\psi'(t) \right]' dt \\ &= \int_1^r \left[t^{-1/2}\psi''(t) - \frac{1}{2}t^{-3/2}\psi'(t) \right] dt, \end{aligned}$$

which holds for each $r > 0$, since $\psi' \in AC_{\text{loc}}(0, \infty)$. Hence, we obtain the following upper bound on the modulus of last integral:

$$\left| \int_1^r t^{-1} \left[t^{1/2}\psi''(t) \right] dt \right| + \frac{1}{2} \left| \int_1^r t^{-1} \left[t^{-1/2}\psi'(t) \right] dt \right|$$

$$\begin{aligned}
&\leq \left| \int_1^r t^{-2} dt \right|^{\frac{1}{2}} \left\{ \left| \int_1^r \left| t^{1/2} \psi''(t) \right|^2 dt \right|^{\frac{1}{2}} + \frac{1}{2} \left| \int_1^r \left| t^{-1/2} \psi'(t) \right|^2 dt \right|^{\frac{1}{2}} \right\} \\
&\leq |1 - r^{-1}|^{\frac{1}{2}} \left\{ \left\| r^{1/2} \psi'' \right\|_{L^2(0, \infty)} + (1/2) \left\| r^{-1/2} \psi' \right\|_{L^2(0, \infty)} \right\}.
\end{aligned}$$

(ii) For the first inequality, we employ the Leibniz-Newton formula

$$r^{-1} \psi(r) - \psi(1) = \int_1^r [t^{-1} \psi(t)]' dt,$$

together with the estimate

$$\begin{aligned}
&\left| \int_1^r t^{-1/2} \left(t^{1/2} [t^{-1} \psi(t)]' \right) dt \right| \\
&\leq \left| \int_1^r t^{-1} dt \right|^{\frac{1}{2}} \left| \int_1^r \left| t^{1/2} [t^{-1} \psi(t)]' \right|^2 dt \right|^{\frac{1}{2}} \\
&\leq |\ln r|^{\frac{1}{2}} \left\| r^{1/2} (r^{-1} \psi)' \right\|_{L^2(0, \infty)},
\end{aligned}$$

the last norm being finite due to $r^{1/2} (r^{-1} \psi)' = r^{-1/2} \psi' - r^{-3/2} \psi$. Since $\psi' \in AC_{\text{loc}}(0, \infty)$, the second inequality is obtained via

$$\psi'(r) - \psi'(1) = \int_1^r \psi''(t) dt,$$

followed by a similar estimate, this time in terms of $\|r^{1/2} \psi''\|_{L^2(0, \infty)}$. \square

2.2 Beppo Levi L_k -splines

As observed in [6], due to the Plancherel-type formula (6), to obtain the variational characterization of the Beppo Levi polyspline s as minimizer of the polar thin plate energy (2), it is sufficient to show that, for each $k \in \mathbb{Z}$, the amplitude coefficient \widehat{s}_k minimizes the corresponding energy component (7). Letting $Q(r, g, g', g'')$ denote the integrand of (7), classical calculus of variations considerations imply that, except at the interpolation locations r_1, \dots, r_n , a minimizer of (7) should satisfy the Euler-Lagrange equation

$$\partial_g Q - \frac{d}{dr} \partial_{g'} Q + \frac{d^2}{dr^2} \partial_{g''} Q = 0.$$

The resulting left-hand side Euler-Lagrange differential operator is given, up to a constant factor, by

$$\begin{aligned}
L_k &:= r \frac{d^4}{dr^4} + 2 \frac{d^3}{dr^3} - \frac{2k^2 + 1}{r} \frac{d^2}{dr^2} + \frac{2k^2 + 1}{r^2} \frac{d}{dr} + \frac{k^4 - 4k^2}{r^3} \\
&= r \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^2} \right)^2.
\end{aligned}$$

Therefore, \widehat{s}_k should necessarily be annihilated by L_k on each subinterval $(0, r_1)$, (r_1, r_2) , \dots , (r_n, ∞) .

The null-space $\text{Ker}L_k$ is computed in [14] via the substitution $r = e^t$, $\frac{d}{dt} = r \frac{d}{dr}$, which transforms L_k into a differential operator with constant coefficients in variable t . Standard factorization then implies

$$L_k = \frac{1}{r^3} \left(r \frac{d}{dr} - |k| \right) \left(r \frac{d}{dr} - |k| - 2 \right) \left(r \frac{d}{dr} + |k| \right) \left(r \frac{d}{dr} + |k| - 2 \right), \quad (10)$$

hence

$$\text{Ker}L_k = \begin{cases} \text{span} \{ r^2, r^2 \ln r, 1, \ln r \}, & \text{if } k = 0, \\ \text{span} \{ r^3, r, r \ln r, r^{-1} \}, & \text{if } |k| = 1, \\ \text{span} \{ r^{|k|+2}, r^{|k|}, r^{-|k|+2}, r^{-|k|} \}, & \text{if } |k| \geq 2. \end{cases} \quad (11)$$

Moreover, the condition that the polar Beppo Levi energy component (7) is finite further restricts the form of \widehat{s}_k on the extreme intervals $(0, r_1)$ and (r_n, ∞) . Specifically, for $k \neq 0$, evaluating (7) for each of the four generating functions of $\text{Ker}L_k$, we obtain the necessary conditions

$$\widehat{s}_k(r) \in \begin{cases} \text{span} \{ r^{|k|+2}, r^{|k|} \}, & \text{for } r \in (0, r_1), \\ \text{span} \{ r^{-|k|+2}, r^{-|k|} \}, & \text{for } r \in (r_n, \infty). \end{cases}$$

Note that $\text{span} \{ r^{|k|+2}, r^{|k|} \} = \text{Ker}G_k$ and $\text{span} \{ r^{-|k|+2}, r^{-|k|} \} = \text{Ker}R_k$, where

$$\begin{aligned} G_k &:= \frac{1}{r} \left[\frac{d^2}{dr^2} - \frac{2|k|+1}{r} \frac{d}{dr} + \frac{|k|(|k|+2)}{r^2} \right] \\ &= \frac{1}{r^3} \left(r \frac{d}{dr} - |k| \right) \left(r \frac{d}{dr} - |k| - 2 \right), \\ R_k &:= \frac{1}{r} \left[\frac{d^2}{dr^2} + \frac{2|k|-1}{r} \frac{d}{dr} + \frac{|k|(|k|-2)}{r^2} \right] \\ &= \frac{1}{r^3} \left(r \frac{d}{dr} + |k| \right) \left(r \frac{d}{dr} + |k| - 2 \right). \end{aligned} \quad (12)$$

Remark 1. It can be verified that r^{-3} is the only factor of the form r^α which, when inserted in front of the last two brackets in the right-hand side of the above formulae, turns G_k and R_k into mutually adjoint operators. Indeed, the formal adjoint of G_k is

$$G_k^* = \frac{d^2}{dr^2} \left(\frac{1}{r} \cdot \right) + (2|k|+1) \frac{d}{dr} \left(\frac{1}{r^2} \cdot \right) + \frac{|k|(|k|+2)}{r^3} = R_k$$

and a similar computation shows $R_k^* = G_k$.

Recall the notation $\rho := \{r_1, \dots, r_n\}$ used in the Introduction.

Definition 1. Let $k \neq 0$. A function $\eta : [0, \infty) \rightarrow \mathbb{C}$ is called a Beppo Levi L_k -spline on ρ if the following conditions hold:

- (i) $L_k \eta(r) = 0, \forall r \in (r_j, r_{j+1}), \forall j \in \{1, \dots, n-1\};$
- (ii) $G_k \eta(r) = 0, \forall r \in (0, r_1),$ and $R_k \eta(r) = 0, \forall r > r_n.$
- (iii) η is C^2 -continuous at each knot $r_1, \dots, r_n.$

The space of all Beppo Levi L_k -splines on ρ will be labelled $\mathcal{S}_k(\rho).$

Due to conditions (ii), $\mathcal{S}_k(\rho)$ is a subspace of Λ_1 if $|k| = 1$, and of Λ_2 if $|k| \geq 2$.

For $k = 0$, the related notion of a Beppo Levi L_0 -spline is treated in [7]. In this case, the correct left/right operators G_0 and R_0 on the extreme intervals are not obtained by just letting $k = 0$ in (12). Also, G_0 and R_0 are not anymore mutually adjoint.

The proof of the next result follows from the definition of biharmonic Beppo Levi polysplines on annuli [5].

Proposition 1. *A univariate function $\eta : [0, \infty) \rightarrow \mathbb{C}$ is a Beppo Levi L_k -spline on ρ , i.e. $\eta \in \mathcal{S}_k(\rho)$, if and only if the polar surface $s(r, \theta) := \eta(r) e^{-ik\theta}$ is a biharmonic Beppo Levi polyspline on annuli determined by ρ .*

We now review some relevant literature. It was pointed out by Kounchev [14, p. 91] that, on any interval of positive real numbers, the null space of L_k can be described as an extended complete Chebyshev (ECT) system in the sense of Karlin and Ziegler [13], via the representation

$$r^{|k|} L_k = D_4 D_3 D_2 D_1 = \frac{d}{dr} r^{2|k|+1} \frac{d}{dr} \frac{1}{r^{2|k|+1}} \frac{d}{dr} r^{2|k|+1} \frac{d}{dr} \frac{1}{r^{|k|}},$$

where $D_1 = \frac{d}{dr} \left(\frac{1}{r^{|k|}} \cdot \right)$, $D_2 = D_4 = \frac{d}{dr} \left(r^{2|k|+1} \cdot \right)$, $D_3 = \frac{d}{dr} \left(\frac{1}{r^{2|k|+1}} \cdot \right)$. We also observe, following Schumaker [20, p. 398], that L_k possesses the factorization

$$L_k = M_k^* M_k, \quad (13)$$

where M_k^* denotes the formal adjoint of

$$\begin{aligned} M_k &:= \frac{1}{\sqrt{r^{2|k|+1}}} \frac{d}{dr} r^{2|k|+1} \frac{d}{dr} \frac{1}{r^{|k|}} = \sqrt{r} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^2} \right) \\ &= r^{-3/2} \left(r \frac{d}{dr} - |k| \right) \left(r \frac{d}{dr} + |k| \right). \end{aligned}$$

Due to (13), a function that satisfies conditions (i) and (iii) of Definition 1 can be characterized as a ‘generalized spline’ or ‘ M_k -spline’ on $[r_1, r_n]$ in the sense of Ahlberg, Nilson, and Walsh [1], Schultz and Varga [19]. However, our labeling such a function as a ‘ L_k -spline’ agrees with the terminology of Lucas [17] and Jerome and Pierce [12], which is more adequate, in view of the fact that L_k may possess other factorizations of the type (13). Indeed, for $k \neq 0$, our adjoint boundary operators G_k and R_k actually generate, via (10), the factorization

$$L_k = G_k r^3 R_k = \tilde{L}_k^* \tilde{L}_k, \quad \text{where } \tilde{L}_k := r^{3/2} R_k. \quad (14)$$

This differs from (13) for $|k| \geq 2$, while it coincides with (13) for $|k| = 1$.

On the other hand, the ‘natural’ end conditions of L -spline literature (see [20]) are always formulated in terms of a single differential operator at both ends of the interpolation domain. It is thus remarkable that adjoint boundary operators as in condition (ii) of our definition have also occurred in [4], in the context of exponential L -splines generated as Fourier coefficients of Beppo Levi polyspline surfaces on parallel strips. Such exponential L -splines coincide in fact with Matérn kernels on the full real line (for Matérn kernels on a compact interval, see [10]). As shown in [3], adjoint L -spline end conditions are intimately connected to Wiener-Hopf factorizations for semi-cardinal interpolation.

3 Interpolation with Beppo Levi L_k -splines

3.1 A fundamental identity

We employ the notations introduced in the previous section.

Theorem 1. (i) *Let $k \in \mathbb{Z}$, $|k| \geq 2$, and an arbitrary Beppo Levi L_k -spline $\eta \in \mathcal{S}_k(\rho)$. Also, assume that $\psi \in \Lambda_2$ vanishes on the knot-set ρ :*

$$\psi(r_j) = 0, \quad \forall j \in \{1, \dots, n\}. \quad (15)$$

Then the following orthogonality relation holds:

$$\int_0^\infty r^3 [R_k \eta(r)] [R_k \bar{\psi}(r)] dr = 0. \quad (16)$$

(ii) *The same conclusion holds if $k = \pm 1$ and $\psi \in \Lambda_1$ satisfies (15).*

Proof. For convenience, let us denote the left-hand side of (16) by $I_k := I_k(\eta, \psi)$. Note that, for any $k \neq 0$, $\eta \in \mathcal{S}_k(\rho)$ implies $R_k \eta(r) = 0$, $\forall r > r_n$, hence we can work with integral I_k on the integration domain $(0, r_n]$. Since

$$r^{3/2} R_k \psi = r^{1/2} \psi'' + (2|k| - 1) r^{-1/2} \psi' + |k|(|k| - 2) r^{-3/2} \psi,$$

the hypotheses imply, via Cauchy-Schwarz inequality, that I_k is an absolutely convergent integral. Using the factorization of the operator R_k and making the notation

$$\begin{aligned} \eta_1(r) &:= \left(r \frac{d}{dr} + |k| - 2\right) \eta(r), \\ \psi_1(r) &:= \left(r \frac{d}{dr} + |k| - 2\right) \psi(r), \end{aligned}$$

we have

$$\begin{aligned} I_k &= \int_0^{r_n} r^{-3} \left[\left(r \frac{d}{dr} + |k|\right) \eta_1(r) \right] \left[\left(r \frac{d}{dr} + |k|\right) \bar{\psi}_1(r) \right] dr \\ &= \sum_{j=1}^n \int_{r_{j-1}}^{r_j} r^{-2} \left[\left(r \frac{d}{dr} + |k|\right) \eta_1(r) \right] \frac{d}{dr} \bar{\psi}_1(r) dr \\ &\quad + \sum_{j=1}^n \int_{r_{j-1}}^{r_j} r^{-3} \left[\left(r \frac{d}{dr} + |k|\right) \eta_1(r) \right] |k| \bar{\psi}_1(r) dr, \end{aligned}$$

where all integrals remain absolutely convergent and $r_0 := 0$. Next, we apply integration by parts in each term of the first sum, which is permitted due to the fact that $\psi_1 \in AC_{\text{loc}}(0, \infty)$. Since

$$\frac{d}{dr} \left\{ r^{-2} \left[\left(r \frac{d}{dr} + |k| \right) \eta_1(r) \right] \right\} = r^{-3} \left[\left(r \frac{d}{dr} - 2 \right) \left(r \frac{d}{dr} + |k| \right) \eta_1(r) \right],$$

we obtain

$$\begin{aligned} I_k &= \sum_{j=1}^n \left[\overline{\psi_1}(r) r^{-2} \left(r \frac{d}{dr} + |k| \right) \eta_1(r) \right]_{r_{j-1}}^{r_j} \\ &\quad - \sum_{j=1}^n \int_{r_{j-1}}^{r_j} r^{-3} \overline{\psi_1}(r) \left(r \frac{d}{dr} - |k| - 2 \right) \left(r \frac{d}{dr} + |k| \right) \eta_1(r) dr. \end{aligned}$$

Since ψ has continuity C^1 and η has continuity C^2 , the first sum of the last display is telescopic, hence we only have to evaluate the boundary terms corresponding to $r := r_n$ and $r := r_0 = 0$. Note that the boundary term at r_n is zero, since the condition $R_k \eta(r) = 0$, $\forall r > r_n$, of a Beppo Levi L_k -spline implies, by continuity, the relation $\left[\left(r \frac{d}{dr} + |k| \right) \eta_1(r) \right]_{r=r_n} = 0$.

For the boundary term at 0, consider first the case $|k| \geq 2$. Then the left end condition $G_k \eta(r) = 0$, *i.e.* $\eta \in \text{span} \{r^{|k|+2}, r^{|k|}\}$, for $r \in (0, r_1)$, implies

$$r^{-2} \left[\left(r \frac{d}{dr} + |k| \right) \eta_1(r) \right] = O(r^{|k|-2}), \quad \text{as } r \rightarrow 0.$$

Since, by Lemma 1, $\psi_1(r) = O(r)$, as $r \rightarrow 0$, we deduce that the boundary term at 0 vanishes if $|k| \geq 2$. If $|k| = 1$, the left end condition implies $\eta \in \text{span} \{r^3, r^1\}$, for $r \in (0, r_1)$, hence

$$r^{-2} \left[\left(r \frac{d}{dr} + 1 \right) \eta_1(r) \right] = cr, \quad \forall r \in (0, r_1),$$

for some constant c . Since, by Lemma 1, in this case $\psi_1(r) = O(r |\ln r|^{1/2})$, as $r \rightarrow 0$, it follows that the boundary term at 0 also vanishes if $|k| = 1$.

On the other hand, for each $j \in \{1, \dots, n\}$, since $\eta \in \text{Ker} L_k$ on the interval (r_{j-1}, r_j) , there exists a constant c_j such that

$$\left(r \frac{d}{dr} - |k| - 2 \right) \left(r \frac{d}{dr} + |k| \right) \eta_1(r) = c_j r^{|k|}, \quad \forall r \in (r_{j-1}, r_j).$$

Hence

$$\begin{aligned} I_k &= \sum_{j=1}^n c_j \int_{r_{j-1}}^{r_j} r^{|k|-3} \left(r \frac{d}{dr} + |k| - 2 \right) \overline{\psi}(r) dr \\ &= \sum_{j=1}^n c_j \int_{r_{j-1}}^{r_j} \frac{d}{dr} \left[r^{|k|-2} \overline{\psi}(r) \right] dr = \sum_{j=1}^n c_j \left[r^{|k|-2} \overline{\psi}(r) \right]_{r_{j-1}}^{r_j}. \end{aligned}$$

For $|k| \geq 2$, since Lemma 1 implies $r^{|k|-2}\bar{\psi}(r) = O(r)$, as $r \rightarrow 0$, and, by hypothesis, $\psi(r_j) = 0, \forall j \in \{1, \dots, n\}$, we deduce $I_k = 0$, as stated. For $|k| = 1$, we reach the same conclusion without the need to investigate $r^{-1}\bar{\psi}(r)$ as $r \rightarrow 0$, since in this case $c_1 = 0$. \square

3.2 Existence, uniqueness, and optimality

Theorem 2. *Let ν_1, \dots, ν_n be arbitrary real values, where $n \geq 2$. For each $k \neq 0$, there exists a unique Beppo Levi L_k -spline $\sigma \in \mathcal{S}_k(\rho)$, such that*

$$\sigma(r_j) = \nu_j, \quad j \in \{1, \dots, n\}. \quad (17)$$

Proof. It is sufficient to prove the existence of a unique function $\tilde{\sigma} \in C^2[r_1, r_n]$ such that $\tilde{\sigma} \in \text{Ker} L_k$ on each subinterval (r_{j-1}, r_j) with $j \in \{2, \dots, n\}$, $\tilde{\sigma}$ satisfies the interpolation conditions (17) in place of σ , and the following endpoint conditions hold:

$$\begin{aligned} \left[\left(r \frac{d}{dr} - |k| \right) \left(r \frac{d}{dr} - |k| - 2 \right) \tilde{\sigma}(r) \right]_{r \rightarrow r_1^+} &= 0, \\ \left[\left(r \frac{d}{dr} + |k| \right) \left(r \frac{d}{dr} + |k| - 2 \right) \tilde{\sigma}(r) \right]_{r \rightarrow r_n^-} &= 0. \end{aligned} \quad (18)$$

Indeed, such a function $\tilde{\sigma}$ can be uniquely extended to the required Beppo Levi L_k -spline $\sigma \in \mathcal{S}_k(\rho)$ by defining

$$\sigma(r) := \begin{cases} c_1 r^{|k|+2} + c_2 r^{|k|}, & \text{if } 0 < r < r_1, \\ \tilde{\sigma}(r), & \text{if } r_1 \leq r \leq r_n, \\ c_3 r^{-|k|+2} + c_4 r^{-|k|}, & \text{if } r_n < r. \end{cases}$$

To verify this, note that c_1 and c_2 (respectively, c_3 and c_4) are uniquely determined by the conditions that σ and σ' are continuous at r_1 (respectively, at r_n). The continuity of σ'' at r_1 and r_n then follows automatically from (18) and from the properties $G_k \sigma(r) = 0, \forall r \in (0, r_1)$, and $R_k \sigma(r) = 0, \forall r > r_n$.

Now, a function $\tilde{\sigma}$ with the properties stated in the previous paragraph is determined by four coefficients on each of the $n - 1$ subintervals (r_{j-1}, r_j) , $j \in \{2, \dots, n\}$. These coefficients are coupled by three C^2 -continuity conditions at each interior knot r_2, \dots, r_{n-1} , the endpoint conditions (18), and the n interpolation conditions (17), which amount to a $4(n - 1) \times 4(n - 1)$ system of linear equations.

To show that this system has a unique solution, we assume zero interpolation data: $\nu_j = 0, j \in \{1, \dots, n\}$. Then the system becomes homogeneous, since the endpoint conditions and the continuity conditions at the interior knots were already homogeneous linear equations. Let $\tilde{\sigma}$ be determined by an arbitrary solution of this homogeneous system and let $\sigma \in \mathcal{S}_k(\rho)$ be the unique extension of $\tilde{\sigma}$ to a Beppo Levi L_k -spline. Taking $\eta = \psi := \sigma$ in (16), we obtain $R_k \sigma(r) = 0$, i.e. $\sigma \in \text{span} \{r^{-|k|+2}, r^{-|k|}\}$, for $r \in (0, \infty)$. Since $\sigma(r_j) = 0, j \in \{1, \dots, n\}$, and $n \geq 2$, we deduce $\sigma \equiv 0$. Therefore the above homogeneous system admits only the trivial solution, which concludes the proof. \square

Theorem 2 also extends to the case $n = 1$. Indeed, for each integer $k \neq 0$, it is straightforward to verify that there exists a unique function φ_k with the properties: $G_k \varphi_k(r) = 0$ for $0 < r < 1$, $R_k \varphi_k(r) = 0$ for $r > 1$, φ_k is C^2 -continuous at $r = 1$, and $\varphi_k(1) = 1$. Its expression

$$\varphi_k(r) = \frac{1}{2} \begin{cases} r^{|k|} [(1 + |k|) + (1 - |k|) r^2], & 0 \leq r \leq 1, \\ r^{-|k|} [(1 - |k|) + (1 + |k|) r^2], & 1 < r, \end{cases} \quad (19)$$

was given in [5, (3.10)] for $|k| \geq 2$ and is also seen to hold for $|k| = 1$. Hence, if $\rho = \{r_1\}$, then $\sigma := \nu_1 \varphi_k(\cdot/r_1)$ is the unique Beppo Levi L_k -spline in $S_k(\rho)$, such that $\sigma(r_1) = \nu_1$. As shown by the next result, if $|k| \geq 2$ and $n \geq 2$, the dilates of φ_k also provide a basis for a linear representation of the interpolant of Theorem 2.

Theorem 3. *Assume that $|k| \geq 2$, $n \geq 2$, and let σ be the Beppo Levi L_k -spline satisfying the interpolation conditions (17) of Theorem 2 for given values ν_1, \dots, ν_n at the knot-set ρ . Then there exist unique coefficients a_1, \dots, a_n , such that*

$$\sigma(r) = \sum_{j=1}^n a_j \varphi_k\left(\frac{r}{r_j}\right), \quad \forall r \geq 0. \quad (20)$$

This result was established in [5, Lemma 3] for the special case in which σ satisfies Lagrange interpolation conditions. The proof given there also applies to our general interpolation conditions (17). Note that representation (20) does not hold for $|k| = 1$, but a similar representation for $k = 0$ appears in [7, Theorem 4].

Remark 2. For $|k| \geq 2$, it was observed in [5] that, if we make the notation

$$\psi_k(t) := e^{-t} \varphi_k(e^t) = \frac{1}{2} e^{-|k||t|} [(1 - |k|) e^{-|t|} + (1 + |k|) e^{|t|}], \quad t \in \mathbb{R},$$

then ψ_k is a positive definite function, due to its positive Fourier transform

$$\widehat{\psi}_k(\tau) = \int_{-\infty}^{\infty} e^{-it\tau} \psi_k(t) dt = \frac{4|k|(k^2 - 1)}{[(|k| - 1)^2 + \tau^2][(|k| + 1)^2 + \tau^2]}, \quad \tau \in \mathbb{R}.$$

This formula shows that ψ_k belongs to the class of generalized Whittle-Matérn-Sobolev kernels recently studied by Bozzini, Rossini, and Schaback [9].

The next result shows that our Beppo Levi L_k -spline interpolants minimize the functional (7), subject to the interpolation conditions.

Theorem 4. *Given $k \neq 0$ and arbitrary real values $\nu_1, \nu_2, \dots, \nu_n$, let σ denote the unique Beppo Levi L_k -spline obtained in Theorem 2. Then $\|\sigma\|_k < \|g\|_k$ whenever g satisfies the same interpolation conditions (17) as σ and $g \neq \sigma$, where $g \in \Lambda_1$ if $|k| = 1$, while $g \in \Lambda_2$ if $|k| \geq 2$.*

Proof. Letting $\eta := \sigma$, $\psi := g - \sigma$, the hypotheses imply that ψ satisfies (15), hence (16) holds by Theorem 1. Since $\psi' \in AC_{\text{loc}}(0, \infty)$ and $\psi \in \Lambda_1$ if $|k| = 1$, while $\psi \in \Lambda_2$ if $|k| \geq 2$, we can use the proof of [5, Formula (5.3)] for $k \neq 0$ to show that

$$\int_0^\infty r^3 [R_k \eta(r)] [R_k \bar{\psi}(r)] dr = \langle \eta, \psi \rangle_k,$$

where

$$\begin{aligned} \langle \eta, \psi \rangle_k := \int_0^\infty & \left\{ \eta'' \bar{\psi}'' + 2k^2 \left[\frac{\eta}{r^2} - \frac{\eta'}{r} \right] \left[\frac{\bar{\psi}}{r^2} - \frac{\bar{\psi}'}{r} \right] \right. \\ & \left. + \left[\frac{k^2 \eta}{r^2} - \frac{\eta'}{r} \right] \left[\frac{k^2 \bar{\psi}}{r^2} - \frac{\bar{\psi}'}{r} \right] \right\} r dr. \end{aligned}$$

Therefore (16) implies the orthogonality property

$$\langle \sigma, g - \sigma \rangle_k = 0,$$

from which

$$\|g\|_k^2 = \|\sigma\|_k^2 + \|g - \sigma\|_k^2, \quad (21)$$

and $\|g\|_k \geq \|\sigma\|_k$, with equality only if $\|g - \sigma\|_k = 0$. The last relation implies $g \equiv \sigma$ if $|k| \geq 2$, since $\|\cdot\|_k$ is a norm in this case. If $|k| = 1$, the semi-norm $\|g - \sigma\|_k$ vanishes if and only if $g(r) - \sigma(r) = ar$, $\forall r \in (0, \infty)$, for some constant a . Since $g - \sigma$ takes zero values at the knots r_1, \dots, r_n , we deduce again $g \equiv \sigma$, which completes the proof. \square

4 Approximation orders

For each $k \neq 0$, the following result establishes L^∞ and L^2 -error bounds for interpolation with Beppo Levi L_k -splines to data functions from Λ_1 or Λ_2 .

Theorem 5. *Let $\rho := \{r_1, \dots, r_n\}$ be a set of nodes with $0 < r_1 < \dots < r_n$, $n \geq 2$, and $h := \max_{1 \leq j \leq n-1} (r_{j+1} - r_j)$. For an integer $k \neq 0$, let $g : (0, \infty) \rightarrow \mathbb{R}$ be a data function such that $g \in \Lambda_1$ if $|k| = 1$, while $g \in \Lambda_2$ if $|k| \geq 2$. Let $\sigma \in \mathcal{S}_k(\rho)$ be the Beppo Levi L_k -spline of Theorem 2, corresponding to data values $\nu_j := g(r_j)$, $1 \leq j \leq n$. Then, for $m \in \{0, 1\}$, we have the error bounds:*

$$\left\| \frac{d^m}{dr^m} (g - \sigma) \right\|_{L^\infty[r_1, r_n]} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{3/2-m} \|g\|_k, \quad (22)$$

$$\left\| \frac{d^m}{dr^m} (g - \sigma) \right\|_{L^2[r_1, r_n]} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{2-m} \|g\|_k. \quad (23)$$

Proof. Similar error bounds for $k = 0$ were obtained in [7, Theorems 5 & 6], along the lines of the classical error analysis for generalized splines [1]. The same arguments are also seen to apply to the present case $k \neq 0$, by replacing the semi-norm $\|\cdot\|_0$ of [7] with $\|\cdot\|_k$ and using the inequality $\int_0^\infty r |g''(r)|^2 dr \leq \|g\|_k^2$, valid for any data function g as in the hypothesis. \square

Remark 3. As in [7], the bounds (22) and (23) also imply an L^p -error bound for $p \in (2, \infty)$. Moreover, a similar analysis to that of [7, Theorem 7] shows that the exponents of h in the above error bounds cannot be increased for the classes \mathcal{A}_1 and \mathcal{A}_2 of data functions.

The main result of this section applies (23) and the corresponding error bound of [7] for $k = 0$ to obtain a L^2 -convergence order for transfinite surface interpolation with biharmonic Beppo Levi polysplines on annuli. To state this result, let W^2 be the Wiener-type algebra of continuous periodic functions $\mu : [-\pi, \pi] \rightarrow \mathbb{R}$ with Fourier coefficients $\hat{\mu}_k$, $k \in \mathbb{Z}$, such that $\sum_{k \in \mathbb{Z}} |\hat{\mu}_k| (1 + |k|)^2 < \infty$. Note that $W^2 \subset C^2[-\pi, \pi]$ and, as observed in [5, Remark 1], any periodic cubic spline belongs to W^2 .

Theorem 6. *Given $F \in C^2(\mathbb{R}^2 - \{0\}) \cap C(\mathbb{R}^2)$ of polar form f such that (2) is finite, assume that $f(r_j, \cdot) \in W^2$ along each domain circle $r = r_j$, $j \in \{1, \dots, n\}$. Let S be either one of the Beppo Levi polysplines S^A or S^B determined in [5, Theorem 1], satisfying the transfinite interpolation conditions (3) for $\mu_j := f(r_j, \cdot)$, $j \in \{1, \dots, n\}$, where also $S^A(0) = F(0)$ and S^B is biharmonic at 0. Then, for $m \in \{0, 1\}$, we have the L^2 -error bound*

$$\left(\int_{r_1}^{r_n} \int_{-\pi}^{\pi} \left| \frac{\partial^m}{\partial r^m} (f - s)(r, \theta) \right|^2 r \, d\theta \, dr \right)^{1/2} \leq 2^{m-1} \sqrt{\frac{r_n}{r_1}} h^{2-m} \|f\|_{BL}. \quad (24)$$

Proof. For each $r \geq 0$, let $\hat{f}_k(r)$, $k \in \mathbb{Z}$, be the Fourier coefficients of $f(r, \theta)$ with respect to θ . The smoothness assumptions on F imply that $\hat{f}_k \in C^2(0, \infty)$, \hat{f}_k is continuous at $r = 0$, $\forall k \in \mathbb{Z}$, and identity (6) holds. Since $\frac{\partial^m}{\partial r^m} (f - s)(r, \cdot) \in C[-\pi, \pi] \subset L^2[-\pi, \pi]$, the following Plancherel formula is also valid for $m \in \{0, 1\}$ and $r \in [r_1, r_n]$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\partial^m}{\partial r^m} (f - s)(r, \theta) \right|^2 d\theta = \sum_{k \in \mathbb{Z}} \left| \frac{d^m}{dr^m} (\hat{f}_k - \hat{s}_k)(r) \right|^2.$$

Moreover, since $\sqrt{r} \frac{\partial^m}{\partial r^m} (f - s) \in C([r_1, r_n] \times [-\pi, \pi]) \subset L^2([r_1, r_n] \times [-\pi, \pi])$, we may multiply the above relation by r and integrate both sides to obtain, via Fubini's theorem,

$$\frac{1}{2\pi} \int_{r_1}^{r_n} \int_{-\pi}^{\pi} \left| \frac{\partial^m}{\partial r^m} (f - s)(r, \theta) \right|^2 r \, d\theta \, dr = \sum_{k \in \mathbb{Z}} \int_{r_1}^{r_n} \left| \frac{d^m}{dr^m} (\hat{f}_k - \hat{s}_k)(r) \right|^2 r \, dr. \quad (25)$$

Note that, for each $j \in \{1, \dots, n\}$, the transfinite interpolation condition $s(r_j, \theta) = f(r_j, \theta)$, $\forall \theta \in [-\pi, \pi]$, is equivalent to $\hat{s}_k(r_j) = \hat{f}_k(r_j)$, $\forall k \in \mathbb{Z}$. Hence, for $k \neq 0$, the error bound (23) implies, for $m \in \{0, 1\}$,

$$\left\| \frac{d^m}{dr^m} (\hat{f}_k - \hat{s}_k) \right\|_{L^2([r_1, r_n])} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{2-m} \|\hat{f}_k\|_k. \quad (26)$$

In addition, it follows from [7, Theorem 6] that this error bound also holds for $k = 0$, since $\widehat{s}_0^A(0) = \widehat{f}_0(0) = F(0)$ and $\widehat{s}_0^B \in \text{span}\{r^2, 1\}$ for $r \in (0, r_1)$.

Therefore (25), (26), and (6) imply

$$\begin{aligned} & \frac{1}{2\pi} \int_{r_1}^{r_n} \int_{-\pi}^{\pi} \left| \frac{\partial^m}{\partial r^m} (f - s^{A,B})(r, \theta) \right|^2 r \, d\theta \, dr \\ & \leq r_n \sum_{k \in \mathbb{Z}} \int_{r_1}^{r_n} \left| \frac{d^m}{dr^m} (\widehat{f}_k - \widehat{s}_k)(r) \right|^2 dr \\ & \leq Ch^{2(2-m)} \sum_{k \in \mathbb{Z}} \left\| \widehat{f}_k \right\|_k^2 = \frac{C}{2\pi} h^{2(2-m)} \|f\|_{BL}^2, \end{aligned}$$

where $C = 2^{2(m-1)} r_n / r_1$, which establishes (24). \square

A similar approximation order for transfinite interpolation via biharmonic Beppo Levi polysplines on parallel strips has recently been proved in [6]. Related Plancherel representations of the error have been employed before by Kounchev and Render [15] for cardinal polysplines on annuli and by Sharon and Dyn [21] for interpolatory subdivision schemes.

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